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Why God Might Play Dice

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We study the question of the existence of hidden variables within the formalism of Pitowsky. We show that probabilities admit factorizable hidden variable models iff they admit a Kolmogorovian representation. In particular, directly deduced experimental frequencies always admit a factorizable hidden variable model and thus a Kolmogorovian representation. We apply this result in the framework of Bell's inequalities. We show that a deterministic interpretation of the hidden variables associated with this situation refutes the possibility for the experimenter of choosing freely the conditions of experimentation.

1. INTRODUCTION

Classical concepts such as determinism, realism, and causality do not easily find a place in the context of quantum mechanics. The intimate belief that such concepts must somehow describe the fundamental laws of nature has motivated the development of numerous hidden variable models. These models try to describe the quantum probability as the weighted average of deterministic truth-values[2] which reflect a hidden order existing at a subquantum level (they describe the dice used by God for deciding what is the result of a quantum measurement). Bell proved in his famous theorem (Bell, 1965) that broad classes of such models generate probabilities which must fulfill some inequalities. He also showed that in some situations quantum probabilities violate these inequalities. More recently, Pitowsky (1989) reformulated the Clauser–Horne inequalities [which are a variant of Bell's inequalities (Clauser and Horne, 1974)] as a necessary and sufficient condition for the existence of a Kolmogorovian representation for these probabilities.

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[2] According to the convention introduced by Pitowsky (Pitowsky, 1989), we mean by the truth-value of an experimental outcome the probability of realization of this outcome. A deterministic truth-value is equal to 0 or 1 by definition.

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We show that the existence of particular hidden variable models (the factorizable ones) for a probability vector is equivalent to the existence of a Kolmogorovian representation for it. We also show a general theorem: we can always find a factorizable hidden variable model allowing us to reproduce any kind of probability which is a directly experimentally deduced frequency (we shall define what this means later). Formulated in the language of Pitowsky, this means that a Kolmogorovian representation always exists for such probabilities. In particular, the effective frequencies related to the so-called Orsay experiments (Aspect *et al.*, 1981) that we shall describe later do not violate the Clauser–Horne inequalities,[3] although it is well known that the quantum frequencies violate them. We show that, in the hidden variable model associated with these effective frequencies, the acts and choices of the experimenter themselves are a quantity determined by the hidden variables (they are fixed by the dice of God).

2. THE EXISTENCE OF A KOLMOGOROVIAN REPRESENTATION AND THE CLASSICAL CORRELATION POLYTOPE

2.1. Some Definitions

• Let us consider a vector $\boldsymbol{\pi}$ in $\mathbf{R}^{n(n+1)/2}$ representing n probabilities related to the realization of n dichotomic properties, and the $n(n-1)/2$ probabilities related to the conjunctions[4] of these properties:

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n, \pi_{12}, \pi_i, \ldots, \pi_{13}, \ldots, \pi_{ij}, \ldots, \pi_{n-1,n}) \qquad (1)$$

• π is said to admit a Kolmogorov representation if there exists a measure μ on a Boolean algebra $\mathcal{A}$ and elements $a_1, \ldots, a_i, \ldots, a_n$ of $\mathcal{A}$ such that

$$\pi_i = \mu(a_i); \qquad \pi_{ij} = \mu(a_i \cap a_j) \qquad (2)$$

• For each n-vector $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_i, \ldots, \epsilon_n) \in \{0, 1\}^n$, $\mathbf{u}^\epsilon$ denotes the following vector of $\mathbf{R}^{n(n+1)/2}$: $(\epsilon_1, \ldots, \epsilon_i, \ldots, \epsilon_n, \epsilon_1\epsilon_2, \epsilon_1\epsilon_3, \ldots, \epsilon_i\epsilon_j, \ldots, \epsilon_{n-1}\epsilon_n)$. The 2^n vectors so defined are called the classical vertices.

• The surface which consists of all the convex linear combinations of these vertices is called the classical polytope:

$$C(n) = \{\mathbf{v} \in \mathbf{R}^{n(n+1)/2}: \mathbf{v} = \sum_{\epsilon \in \{0,1\}^n} \lambda_\epsilon \mathbf{u}^\epsilon, \lambda_\epsilon \geq 0, \sum_{\epsilon \in \{0,1\}^n} \lambda_\epsilon \leq 1\} \qquad (3)$$

The following theorem due to Pitowsky (1989) allows us to express the

[3] An example of this property appeared originally in a preprint of Szabo (1995).
[4] Later, we shall define more precisely what we intend by the conjunction of properties.

existence of a Kolmogorovian representation of the probability π as a geometrical condition in $\mathbf{R}^{n(n+1)/2}$.

Theorem 1. The vector π in $\mathbf{R}^{n(n+1)/2}$ defined above admits a Kolmogorovian representation iff it belongs to the classical polytope.

3. THE POSSIBILITY OF HIDDEN VARIABLES

Let us consider a system consisting of an urn containing different balls of diverse colors and masses. We could carry out an experiment by taking one ball "at random" among the balls of the urn and observing the color and mass of this ball (this is a variant of the dice). The frequency of occurrence of balls of one particular color (mass) would be, when the balls are homogeneously mixed,[5] the ratio of the amount of balls sharing this color (mass) to the total amount of balls. The frequency of occurrence of balls of one particular color and one particular mass would be, when numerous experiments are carried out, very close to the ratio of the amount of balls sharing this color and this mass to the total amount of balls. We shall consider this experiment as a typical example where "hidden variables" appear. We intend thereby that, although each individual experiment is deterministic (unambiguous), our lack of knowledge (or eventually of control) of the experimental conditions (or of the prepared state of the system, here of the urn) leads to a probabilistic behavior. This justifies the following definition:

Definition. A probability vector π admits a hidden variable model iff:

- We can find a set of variables such that each variable determines univoquely each result of the $n(n+1)/2$ properties associated to the $n(n+1)/2$ probabilities contained in the vector π; this means that to each variable we can associate a truth-function on the set of properties, equal to 1 when a property is realized, 0 otherwise, so to say an element of $\{0, 1\}^{n(n+1)/2}$.
- The frequency of occurrence of these hidden variables is given by a measure P on the set of hidden variables.
- The frequency of realization of the $n(n+1)/2$ properties obtained by averaging with the weight P their truth-functions on the set of hidden variables is equal to the vector π.

This definition is more than needed, as is shown by the following theorem.

[5] It seems through this example that a hidden variable model contains necessary stochasticity as an essential tool, but this stochasticity itself could be the consequence of some complicated but deterministic dynamics. Such dynamics are, for instance, used in computers, in programs aimed at generating random numbers.

Theorem 2. Each probability vector admits a hidden variable model.

Proof. We can without loss of generality redefine the order of the $n(n + 1)/2$ probabilities contained in the vector $\boldsymbol{\pi}$ so that they are increasing in function of the index i $\{i \in [1, n(n + 1)/2]\}$. If we choose as hidden variables the $n(n + 1)/2$ variables v_i $\{i \in [1, \ldots, n(n + 1)/2]\}$ of which the truth-value is zero for all the experiments associated to an index strictly smaller than i and one otherwise, and that the frequency of occurrence P_i of the variable v_i is the difference between the ith component of the reordered vector $\boldsymbol{\pi}$ and the foregoing component (we suppose for convenience that the frequency of occurrence of the first component of the reordered vector equals this component), we fulfill the conditions under which $\boldsymbol{\pi}$ admits a hidden variable model.

It is clear that the model that we introduced in the proof contains more information than the $n(n + 1)/2$ probabilities of the vector $\boldsymbol{\pi}$. It implies, for instance, that if we could observe simultaneously the realization of two of the $n(n + 1)/2$ properties associated to the probabilities, the realization of the property which has the smallest probability would always imply the realization of the other property. Such correlations may not be realized experimentally. This hidden variable model is then not adequate. Furthermore, the choice of notation π_{ij} made by Pitowsky reflects implicitly that this probability corresponds to the simultaneous realization of the property related to the probability π_i and of the one related to π_j, and the hidden variable model presented above ignores the correlation implied by this implicit convention. This justifies the following definition:

Definition. A probability vector $\boldsymbol{\pi}$ admits a factorizable hidden variable model iff:

- It admits a hidden variable model.
- The truth-function associated to each hidden variable is factorizable: the truth-value of the result described by the index ij is the product of the truth-value of the ith result with the truth-value of the jth result.

This last condition expresses that, if the hidden variable model is physically relevant, whenever we observe the ith result and the jth result simultaneously, then we observe the result associated to the index ij. Implicitly, this means that these two observations are compatible (we can carry them out simultaneously), and that what we called initially the conjunction of the events i and j is effectively the simultaneous realization of these events. This condition restricts the set of admissible probabilities, as shown by the following theorem:

Theorem 3. A vector $\boldsymbol{\pi}$ in $\mathbf{R}^{n(n+1)/2}$ admits a factorizable hidden variable model iff it belongs to the classical polytope.

Proof. (A) Since the truth-function associated to a hidden variable is factorizable, it can be written as a classical vertex of $\mathbf{R}^{n(n+1)/2}$ of the kind $\mathbf{u}^{\epsilon}$, $\epsilon \in \{0, 1\}^n$, as was defined in the first section. According to the definition of a hidden variable model, $\boldsymbol{\pi} = \Sigma_{\epsilon \in \{0,1\}^n} \mathbf{u}^{\epsilon} \cdot P(\epsilon)$, where $P(\epsilon)$ is the measure of the set of hidden variables admitting $\mathbf{u}^{\epsilon}$ as truth-function. These sets are disjoint for distinct truth-functions, and they cover the set of hidden variables, because each hidden variable determines a factorizable truth-function. Taking into account that P is a measure, we have thus that the vector $\boldsymbol{\pi}$ is a convex linear combination of the classical vertices.

(B) If the vector $\boldsymbol{\pi}$ is a convex linear combination of the classical vertices, we can define the following hidden variable model:

- Each element ϵ of $\{0, 1\}^n$ is a hidden variable, and its truth-function is $\mathbf{u}^{\epsilon}$.
- The measure $P(\epsilon)$ is equal to the weight λ^{ϵ} of $\mathbf{u}^{\epsilon}$ in the convex linear combination when ϵ is not 0^n.
- Then, its weight is equal to 1 − the sum of the weights of the other hidden variables.

It is easy to check that P is a measure, and that this model is a factorizable hidden variable model of $\boldsymbol{\pi}$, which finishes the proof.

4. THE PROBABILITIES AS EXPERIMENTAL FREQUENCIES

The probabilities which appear in physical theories can in general be measured experimentally as frequencies of occurrence of events. This is the case, for instance, in scattering experiments, where counters record the number of realizations of some given event and deduce its probability by dividing this number by the total number of events. The probability vector

$$(\pi_1, \pi_2, \ldots, \pi_i, \ldots, \pi_n, \pi_{12}, \pi_{13}, \ldots, \pi_{ij}, \ldots, \pi_{n-1,n})$$

is then equal to

$$(N_1, N_2, \ldots, N_i, \ldots, N_{12}, N_{13}, \ldots, N_{ij}, \ldots, N_{n-1,n}) \cdot (1/N_T)$$

where N_x represents the number of events accompanied by the realization of the dichotomic property x during the period T of an experimental run, N_T being the total number of events occurring during this experimental run.

This justifies the following definition:

Definition. If we can simultaneously observe the realization (nonrealization) of the n properties symbolized by the index i, and we associate the index ij to the simultaneous realization of the properties i and j, we shall say that the vector $\boldsymbol{\pi}$ is directly experimentally deduced.

In this case, we can show the following theorem:

Theorem 4. If the vector $\boldsymbol{\pi}$ is directly experimentally deduced, it admits a factorizable hidden variable model.

Proof. Let us define the following hidden variable model:

- Each experimental result associated to an event of the run can be expressed as an element ϵ of $\{0, 1\}^n$, where ϵ_i is the result of the ith property (1 when it is realized, 0 otherwise) and we shall consider it as a hidden variable (its truth-function is $\mathbf{u}^\epsilon$).
- The measure $P(\epsilon)$ of each hidden variable is equal to N_ϵ/N_T, where N_ϵ is the number of events realizing the ith property whenever ϵ_i is one, and not realizing it otherwise.

It is easy to check that P is a measure, and that this model is a factorizable hidden variable model of $\boldsymbol{\pi}$, which finishes the proof.

5. HIDDEN VARIABLE MODELS AND DETERMINISM

It is important to note that the possibility of a Kolmogorovian representation and thus of a factorizable hidden variable model does not per se imply the possibility of a deterministic mechanism explaining the observations. Nevertheless this identification is true in nearly all cases: a Kolmogorovian probability can "nearly always" (we intend by this to except extremely pathological cases) be deduced from a deterministic model. The dynamics of complex systems such as, for instance, the baker's transformation allows one to generate a homogeneous probability distribution over the interval [0, 1] (this is in connection with footnote 4). After a mapping of this interval, we can then generate arbitrary distributions on a finite set of events, or on sufficiently regular continuous intervals. Combining this with the fact that, practically, nearly all experiments possess a finite number of possible results or at worse regular continuous intervals of them and that all the frequencies which are directly experimentally deduced admit a factorizable hidden variable model, there exists always a deterministic model explaining all the facts that we could observe.

Kant already noted, in a more philosophical context, that the answers to some questions are just a matter of belief. The question of whether the universe is a machine is of the same nature. No experiment could refute this assertion. Furthermore, different interpretations are possible concerning the origin of the existence of probabilities. They could be related to a complex dynamical system[6] possessing iteration properties similar to the baker's transformation, or they could be *a priori* probabilities. The appearance of both kinds of probability could occur at the level of the system itself, or during the process of measurement. In the first case, we could speak of a lack of control of the system; in the second case, we could speak of a lack of knowledge.

6. SUMMARY

In order to be more synthetic, we give here a summary of the foregoing theorems:

The probability vector $\boldsymbol{\pi}$ belongs to the classical correlation polytope.

$\Updownarrow$

It admits a Kolmogorovian representation.

$\Updownarrow$

It admits a factorizable hidden variable model.

$\Uparrow$

It is directly experimentally deduced.

All probability vectors admit a hidden variable model, not necessarily factorizable.

7. CLAUSER–HORNE INEQUALITIES AND THE CLASSICAL POLYTOPE

7.1. Clauser–Horne Inequalities

The experiments realized to test Bell's inequalities (Bell, 1965) proceed as follows. A source emits two photons quasisimultaneously. Two polarizers are placed in two spatially separated regions (left and right) symmetrically on both sides of the source. They allow one to measure a dichotomic variable, the sign of the linear polarization of the incoming photons along a direction belonging to the plane perpendicular to its direction of propagation. At each side, we choose at our convenience a direction for the measurement of polarization between two different directions: the directions A and A' in the left region, B and B' in the right region. We associate to the property "the photon has + polarization" the value 1, 0 otherwise. The conjunction of these experiments is then expressed by the product of the values of each experiment. The technical details are not important here, but it is worth noting

[6] Thanks to S. Diner for clarifying discussions about this subject.

that for some good chosen directions of the polarizers[7] we obtain by orthodox quantum mechanical computations that the probabilities ($P(A)$, $P(A')$, $P(B)$, $P(B')$, $P(A \cap B)$, $P(A \cap B')$, $P(A' \cap B)$, $P(A' \cap B')$) yield

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{\sin^2(\pi/8)}{2}, \frac{\sin^2(\pi/8)}{2}, \frac{\sin^2(3\pi/8)}{2}, \frac{\sin^2(\pi/8)}{2}\right)$$

where X, $(X \cap Y)$ represent the property "the photon has + polarization along the X direction, (along the X and Y directions simultaneously)." These probabilities were observed as experimental frequencies with a very good precision in numerous experiments,[8] among others in the Orsay experiments (Aspect *et al.*, 1981).

It is worth remarking that the probabilities $P(A' \cap A)$ and $P(B \cap B')$ are not taken into account, because the choice of a direction for a polarizer excludes the other direction, so that we cannot simultaneously measure A and A' (B and B'). Note that this implies that the frequencies deduced from the experiments are not directly experimentally deduced in the sense of our previous definition.

If there exists a hidden variable model which is factorizable only for the four conjunctions considered, the same proof as in the general case (with six conjunctions) is still valid and implies that the eight-dimensional probability vector deduced from the experiment is a convex combination of the 16 (2^4) classical vertices which are obtained by suppressing the pair of nonphysical components of the ten [$n(n + 1)/2$, $n = 4$]-dimensional vertices already defined. Pitowsky (1989) showed that we can rewrite this condition in the form of inequalities:

Theorem 5. The probability vector ($P(A)$, $P(A')$, $P(B)$, $P(B')$, $P(A \cap B)$, $P(A \cap B')$, $P(A' \cap B)$, $P(A' \cap B')$) [it is equal to (1/2, 1/2, 1/2, 1/2, $(1 - \sqrt{2}/2)/4$, $(1 - \sqrt{2}/2)/4$, $(1 + \sqrt{2}/2)/4$, $(1 - \sqrt{2}/2)/4$) in our case] is a convex combination of the reduced classical vertices (ϵ_1, ϵ_2, ϵ_3, ϵ_4, $\epsilon_1\epsilon_3$, $\epsilon_1\epsilon_4$, $\epsilon_2\epsilon_3$, $\epsilon_2\epsilon_4$) (where $\epsilon_i \in \{0, 1\}\ \forall i$) iff the following inequalities are fulfilled:

$$-1 \leq P(A \cap B) + P(A \cap B') + P(A' \cap B') - P(A' \cap B) - P(A) - P(B') \leq 0 \quad (4)$$

$$-1 \leq P(A' \cap B) + P(A' \cap B') + P(A \cap B') - P(A \cap B) - P(A') - P(B') \leq 0 \quad (5)$$

[7] The source emits a pair of photons forming an entangled state describable by the singlet state, the directions A, A', B, B' are coplanar and are all separated by angles of 22.5 deg, in the order A', B', A, B.

[8] Provided we make some quite natural assumptions about the quality and the functioning of the polarizers. Some physicists [see Home and Selleri (1991) for a review of the polemics] still contest the experiments, considering that the quality of the photon detectors is too low to deduce valid results. We postulate here that we can trust the experimenters and the assumptions that they made about the behavior of the detectors.

$$-1 \leq P(A \cap B') + P(A \cap B) + P(A' \cap B) - P(A' \cap B') - P(A) - P(B) \leq 0 \quad (6)$$

$$-1 \leq P(A' \cap B') + P(A' \cap B) + P(A \cap B) - P(A \cap B') - P(A') - P(B) \leq 0 \quad (7)$$

$$0 \leq P(X, Y) \leq P(X) \leq 1 \quad (8)$$

$$0 \leq P(X, Y) \leq P(Y) \leq 1 \quad (9)$$

$$P(X) + P(Y) - P(X \cap Y) \leq 1 \quad (10)$$

where

$$X \in \{A, A'\}, \qquad Y \in \{B, B'\} \quad (11)$$

The first four inequalities are known as the Clauser–Horne (Clauser and Horne, 1971) inequalities. It is easy to show the necessary condition, because every reduced classical vertex fulfills the inequalities and these inequalities are linear. The sufficient condition is less straightforward, and we invite the interested reader to consult the book of Pitowsky.

If we replace the experimentally measured frequencies in the first inequality, we violate it:

$$\frac{1 - \sqrt{2}/2}{4} + \frac{1 - \sqrt{2}/2}{4} + \frac{1 - \sqrt{2}/2}{4} - \frac{1 + \sqrt{2}/2}{4} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2} - \frac{\sqrt{2}}{2} < -1$$

This means that the probability vector associated to the measures of correlation in this experiment does not admit a Kolmogorovian representation.

7.2. Effective Probabilities Do Not Violate the Inequalities

We showed that a probability vector which is directly experimentally deduced admits a factorizable hidden variable model in Theorem 4, and that the violation of Clauser–Horne inequalities implies the nonexistence of such a model in Theorem 5. The violation of these inequalities by experimental frequencies seems paradoxical unless we notice that the frequencies (1/2, 1/2, 1/2, 1/2, $(1 - \sqrt{2}/2)/4$, $(1 - \sqrt{2}/2)/4$, $(1 + \sqrt{2}/2)/4$, $(1 - \sqrt{2}/2)/4$) are not really observed. Reformulated according to our definitions, this means that this probability vector is *not* directly experimentally deduced. How could this be so? The answer is the following. We noticed already that it is impossible to perform simultaneously A and A' (B and B'). In fact the projectors associated to these quantum measurements do not commute. This means that we deduced the frequencies from four different experiments, $(A \cap B)$, $(A \cap B')$, $(A' \cap B)$, $(A' \cap B')$, and not directly from one experiment. We could reformulate these four experiments as constituting one big experiment. For instance, in Orsay experiments (Aspect *et al.*, 1981), the directions of the polarizers were themselves monitored by an electronic device possessing stochastic elements, so that the probability of measuring the polarization along the $A(A')$ direction is 1/2 (the same is true for B and B'). The effective frequency vector is then

$$\left(\frac{1}{2}\cdot\frac{1}{2}, \frac{1}{2}\cdot\frac{1}{2}, \frac{1}{2}\cdot\frac{1}{2}, \frac{1}{2}\cdot\frac{1}{2}, \frac{1-\sqrt{2}/2}{4}\cdot\frac{1}{4}, \frac{1-\sqrt{2}/2}{4}\cdot\frac{1}{4}, \frac{1+\sqrt{2}/2}{4}\cdot\frac{1}{4}, \frac{1-\sqrt{2}/2}{4}\cdot\frac{1}{4}\right)$$

which does not violate the Clauser–Horne inequalities[9]:

$$\frac{1-\sqrt{2}/2}{4}\cdot\frac{1}{4} + \frac{1-\sqrt{2}/2}{4}\cdot\frac{1}{4} + \frac{1-\sqrt{2}/2}{4}\cdot\frac{1}{4} - \frac{1+\sqrt{2}/2}{4}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2} = -\frac{3}{8} - \frac{\sqrt{2}}{8} > -1$$

The nonviolation of the inequalities in the particular case considered here can be understood as a consequence of the general Theorems 1, 4, and 5, because the effective frequencies correspond to directly experimentally deduced frequencies.

Similar results hold in numerous situations. For instance, if the probability of measuring the polarization along the $A(A')$ direction is not 1/2, or if the directions of the polarizers are changed, the effective frequencies that we obtain still fulfill the inequalities. We developed in Durt (1995) a rather technical proof adapted to this situation, making use of the classical polytope associated to the Orsay experiment. We give in Durt (1996a, b) more general proofs, which allow us to determine explicitly the Kolmogorovian representation of vectors obtained as weighted averages of one-run quantum frequencies.

In the case of the Orsay experiments, the four Kolmogorovian representations of the probabilities characterizing the measurements of polarization along the directions (A, B), (A, B'), (A', B), and (A', B') are given in Table I.[10] We adopted the following convention: the symbol A $(\neg A)$, for instance, is associated with the measurement of a positive (negative) polarization when the left polarizer is oriented along the direction A.

If we consider the effective frequencies associated with the whole experiment, their Kolmogorovian representation is shown in Table II.

7.3. A Hidden Variable Model for the Effective Frequencies

The Kolmogorovian representation given in Table II and Theorem 3 allow us to build a hidden variable model for the effective probabilities. The

[9] This was already noticed by Szabo (1995). We present it here as a special case of the general theorems previously demonstrated.
[10] For the deduction of Tables I and II, see Durt (1996b).

Table I. The Kolmogorovian Representations Associated with the Four One-Run Experiments Entering the Orsay Experiment

$a \cap b$		$a \cap b'$	
$A \cap B$	$A \cap \neg B$	$A \cap B'$	$A \cap \neg B'$
$\frac{1 - \sqrt{2}/2}{4}$	$\frac{1 + \sqrt{2}/2}{4}$	$\frac{1 - \sqrt{2}/2}{4}$	$\frac{1 + \sqrt{2}/2}{4}$
$\neg A \cap B$	$\neg A \cap \neg B$	$\neg A \cap B'$	$\neg A \cap \neg B'$
$\frac{1 + \sqrt{2}/2}{4}$	$\frac{1 - \sqrt{2}/2}{4}$	$\frac{1 + \sqrt{2}/2}{4}$	$\frac{1 - \sqrt{2}/2}{4}$
$a' \cap b$		$a' \cap b'$	
$A' \cap B$	$A' \cap \neg B$	$A' \cap B'$	$A' \cap \neg B'$
$\frac{1 + \sqrt{2}/2}{4}$	$\frac{1 - \sqrt{2}/2}{4}$	$\frac{1 - \sqrt{2}/2}{4}$	$\frac{1 + \sqrt{2}/2}{4}$
$\neg A' \cap B$	$\neg A' \cap \neg B$	$\neg A' \cap B'$	$\neg A' \cap \neg B'$
$\frac{1 - \sqrt{2}/2}{4}$	$\frac{1 + \sqrt{2}/2}{4}$	$\frac{1 + \sqrt{2}/2}{4}$	$\frac{1 - \sqrt{2}/2}{4}$

Table II. The Kolmogorovian Representation Associated with the Whole Orsay Experiment

$A \cap B$	$A \cap \neg B$	$A \cap B'$	$A \cap \neg B'$
$\frac{1 - \sqrt{2}/2}{16}$	$\frac{1 + \sqrt{2}/2}{16}$	$\frac{1 - \sqrt{2}/2}{16}$	$\frac{1 + \sqrt{2}/2}{16}$
$\neg A \cap B$	$\neg A \cap \neg B$	$\neg A \cap B'$	$\neg A \cap \neg B'$
$\frac{1 + \sqrt{2}/2}{16}$	$\frac{1 - \sqrt{2}/2}{16}$	$\frac{1 + \sqrt{2}/2}{16}$	$\frac{1 - \sqrt{2}/2}{16}$
$A' \cap B$	$A' \cap \neg B$	$A' \cap B'$	$A' \cap \neg B$
$\frac{1 + \sqrt{2}/2}{16}$	$\frac{1 - \sqrt{2}/2}{16}$	$\frac{1 - \sqrt{2}/2}{16}$	$\frac{1 + \sqrt{2}/2}{16}$
$\neg A' \cap B$	$\neg A' \cap \neg B$	$\neg A' \cap B'$	$\neg A' \cap \neg B'$
$\frac{1 - \sqrt{2}/2}{16}$	$\frac{1 + \sqrt{2}/2}{16}$	$\frac{1 + \sqrt{2}/2}{16}$	$\frac{1 - \sqrt{2}/2}{16}$

specification of a hidden variable must determine univocally the result of an experiment. What are these results in our case? There are four possible experiments: $(A \cap B)$, $(A \cap B')$, $(A' \cap B)$, $(A' \cap B')$, each of them having four possible results: (up, up), (down, down), (down, up), (up, down).

The classical vertex of the hidden variable predicting the result (up, up) for the first experiment is, for instance, (1, 0, 1, 0, 1, 0, 0, 0), and

$$\left(\frac{1}{2}\cdot\frac{1}{2}, \frac{1}{2}\cdot\frac{1}{2}, \frac{1}{2}\cdot\frac{1}{2}, \frac{1}{2}\cdot\frac{1}{2}, \frac{1 - \sqrt{2}/2}{4}\cdot\frac{1}{4}, \frac{1 - \sqrt{2}/2}{4}\cdot\frac{1}{4}, \frac{1 + \sqrt{2}/2}{4}\cdot\frac{1}{4}, \frac{1 - \sqrt{2}/2}{4}\cdot\frac{1}{4}\right)$$

is a convex combination of such vertices. Every hidden variable determines such a vertex, and determines thus the direction of polarization chosen during the stochastic process monitoring this direction. We could imagine that this process makes use of electronics, or even of a freely thinking physicist choosing at random the directions of the polarizers. Then our hidden variable would predetermine[11] the result obtained by the freely thinking physicist choosing at random. Such a global determinism is in fact a logically coherent explanation of all the apparently hazardous events occurring in the world and the belief in such a determinism is an old psychological attitude commonly called "fatalism." We have shown in fact nothing else than trivial evidence: it is impossible to demonstrate by experiments whether all the events that we observe were already "written in a book" or not unless we know the book in advance. This remark is in some way contradictory with some claims made in the literature about the possibility of a Kolmogorovian representation for quantum probabilities, which we discuss in more detail in the next section.

8. THE POSSIBILITY OF A KOLMOGOROVIAN REPRESENTATION FOR QUANTUM PROBABILITIES

Pitowsky showed (Pitowsky, 1989) that, in a Bell-like situation, the Clauser–Horne inequalities are a necessary and sufficient condition for the existence of a Kolmogorovian representation of the probabilities. According to some authors (Accardi, 1984; Gudder, 1984; Pitowsky, 1982; see Szabo, 1994, for review), the experimental violation of Bell's inequalities could mean that the axioms of probability defined by Kolmogorov are not fulfilled in nature. This is a "negative explanation" of nature in the same way that the Michelson–Morley experiment could be interpreted as the proof of the nonexistence of the ether.

The example given by L. Szabo showed that, provided we consider the effective probabilities, the Bell inequalities are no longer violated. This allows us to build a Kolmogorovian representation of the observed frequencies, in apparent contradiction with the negative conclusion presented above. The essential novelty introduced by Szabo is to consider effective frequencies and not one-run quantum frequencies [note that a prototype of this idea appeared already in Aerts (1987)].

It is worth noticing that the approach emphasized in our paper, centered on the concept of experimentally deduced frequency, not only generalizes

[11] We interpret here the existence of a Kolmogorovian representation (and thus of a hidden variable model) of the probabilities in the sense of the existence of a deterministic machinery simulating the observed frequencies. This is a subjective interpretation of the mathematical results so far obtained, motivated by the general interest of such questions as free will and determinism. This remark is to be connected with Section 5.

the example of Szabo, it constitutes in fact a return to the foundations of the theory of probabilities. The original formulation of probabilities was effectively based on individual equiprobable events. It was much later that Kolmogorov introduced his abstract definition of probabilities (as measures on σ-algebras). It is only recently that, following such authors as Accardi and Pitowsky, it was suggested that the axioms of Kolmogorov are not fulfilled in nature. When we demonstrated that experimentally deduced frequencies admit a Kolmogorovian representation, we used the original formulation of probabilities in terms of equiprobable events. Our approach privileges in fact good sense rather than mathematical refinements. Nevertheless, the interpretation of our results is disturbing: it is usually considered as scientific dogma that the conditions of experiment are freely chosen by the experimenter. It is interesting to remark that the formalism developed by Pitowsky allows one to discuss the question of determinism via rigorous mathematical arguments, as did the formalism of Bell for the question of locality.[12]

APPENDIX. (NON)FACTORIZABILITY AND (NON)LOCALITY

Some authors claim that Bell's inequalities have nothing to do with the problem of nonlocality [see Szabo (1994) for a complete discussion], but are only related to the possibility of a Kolmogorovian representation for the quantum probabilities. We analyze in Durt (1995a, b) the role of the implicit assumption of factorizability hidden in the definition of a Kolmogorovian representation. We show there that locality (or, more generally, separability), as conceived in the original formulation of Bell's theorem, is still present in the formulation of the problem in terms of axiomatic probability. Effectively, a local hidden variable model (in fact a model describing separated entities in the left and right regions) is necessarily factorizable. To show this, let us assume that the pair of photons is in a given hidden state and that no connection at all exists between the left and right regions.[13] The results of the measurements in the left and the right regions are thus predetermined and independent of the direction of detection of polarization at the other side.

[12] We show in Durt (1995a, b) that the hidden variable model of Bohm (1952), when applied to a Bell-like situation, is not local and not factorizable, in accordance with the results of the Appendix, where we show that locality implies factorizability. This allows one to develop an interpretation where the experimenter is free to decide which experiments to perform, but where the left and right polarizers are nonlocally connected, and exchange information in such a way that the results of the polarizers are no longer factorizable.

[13] An essential characteristic of the Orsay experiments is that the choices of the directions of the polarizers were realized by two stochastic devices, supposed to be independent, so quickly that the particles could not receive or exchange information about the direction of polarization realized at the opposite wing of the device, unless this information propagates faster than light. The combination of nonseparability and impossibility of exchanging information without breaking Einsteinian causality was called nonlocality.

This means that the value of the property $A \cap B$ is equal to the product of the values of the properties A and B considered separately. This is nothing else than factorizability. Note that this reasoning was already present in the original formulation of Bell's theorem.

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REFERENCES

Accardi, L. (1984). The probabilistic roots of the quantum mechanical paradoxes, in *The Wave–Particle Dualism*, S. Diner *et al.*, eds., Reidel, Dordrecht.

Aerts, D. (1987). The description of separated systems in quantum mechanics and a possible explanation for the probabilities of quantum mechanics, in *Microphysical Reality and Quantum Formalism*, A. van der Merwe *et al.*, Kluwer, Dordrecht.

Aspect, A., Dalibard, P., and Roger, G. (1981). Experimental tests of realistic local theories via Bell's theorem, *Physical Review Letters*, **47**, 460.

Bell, J. S. (1964). On the EPR paradox, *Physics*, **1**, 195.

Bohm, D. (1952). A suggested interpretation of quantum theory in terms of hidden variables, *Physical Review*, **85**, 166.

Clauser, J. F., and Horne, M. A. (1974). Experimental consequences of objective local theories, *Physical Review D*, **10**, 526.

Durt, T. (1995). Three interpretations of the violation of Bell's inequalities, *Foundations of Physics*, submitted.

Durt, T. (1996a). From quantum to classical, a toy model, Doctoral thesis (January 1996).

Durt, T. (1996b). Proof of the existence of a Kolmogorovian representation for effective frequencies, Preprint TENA, VUB.

Home, D., and Selleri, F. (1991). Bell's theorem and the EPR paradox, *Nuovo Cimento*, **14**(9), 1–98.

Gudder, S. P. (1984). Reality, locality, and probability, *Foundations of Physics*, **14**(10), 997–1011.

Pitowsky, I. (1982). Resolution of the EPR and Bell paradoxes, *Physical Review Letters*, **48**, 1299.

Pitowsky, I. (1989). *Quantum Probability. Quantum Logic*, Springer-Verlag, Berlin.

Szabo, L. (1994). On the real meaning of Bell's theorem, *Journal of Theoretical Physics*, **33**, 191.

Szabo, L. (1995). Is quantum mechanics compatible with a deterministic universe? Two interpretations of quantum probabilities, *Foundations of Physics Letters*, **8**, 421.